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# Shallow water waves on shear flows

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An equation for waves on the surface of a flow with shear is deduced and shown to reduce by suitable scaling to the classical equation of Korteweg & de Vries, which describes such motions on a stationary flow. For steady flows the corresponding theory of cnoidal waves is obtained and the results of Benjamin (1962) for a solitary wave recovered.

## 1. Introduction

Several years ago, Benjamin (1962) showed that it was possible to generalize the classical solitary wave theory (Lamb 1953, ch. 3, pp. 423–427) to waves on moving water with shear. Recent developments in the theory of asymptotic matching have shown how to place the classical cnoidal wave theory, of which the solitary wave is a particular case, in the context of far field solutions of linearized shallow water wave theory (Cole 1968, ch. 5, pp. 248–256). A scaling can be developed which reduces the equations of motion in the far field limit to a single equation, the Korteweg–de Vries equation (Korteweg & de Vries 1895).

In the present paper, it will be shown how it is possible to do this for waves arising on a flow with shear. In the particular case of steady flow a generalized theory of cnoidal waves is deduced which in one case gives the solitary wave deduced by Benjamin (1962).

The original linearized theory of long waves on shear flow was given by Burns (1953), where it was shown that the wave propagation speeds $C$ on a shear flow $u = U(y)$ are given by

$$\int_0^h \frac{dy}{(U(y)-C)^2} = \frac{1}{g}, \tag{1.1}$$

where $h$ is the depth.

To obtain a far-field theory it is necessary to consider perturbations to the equations of motion for large times when the time rates of change are small compared with the rates of change in co-ordinates moving with the wave. The resulting equations are then non-linear, although by suitable manipulation they can be reduced to one equation for the variation of the surface elevation. Once a solution of this equation has been obtained the variation of the flow parameters over the rest of the flow field can be deduced. The solutions of this Korteweg–de Vries equation are now well understood from the work of Kruskal and others (see Zabusky 1967), but reduction to tractable analytic results appears possible only for steady motions.

## 2. The equations of motion

The wave motion to be discussed will be assumed to occur in two dimensions on an inviscid incompressible fluid. It is convenient to start from the unsteady equations of motion in two dimensions for conservation of momentum and mass. Space co-ordinates $x'$ and $y'$ will be chosen parallel and perpendicular to the surface of the undisturbed flow which is assumed to have a velocity in the $x'$ direction $U'(y')$. The undisturbed surface of the flow will be denoted by $y' = h$ and the bottom by $y' = 0$ (figure 1).

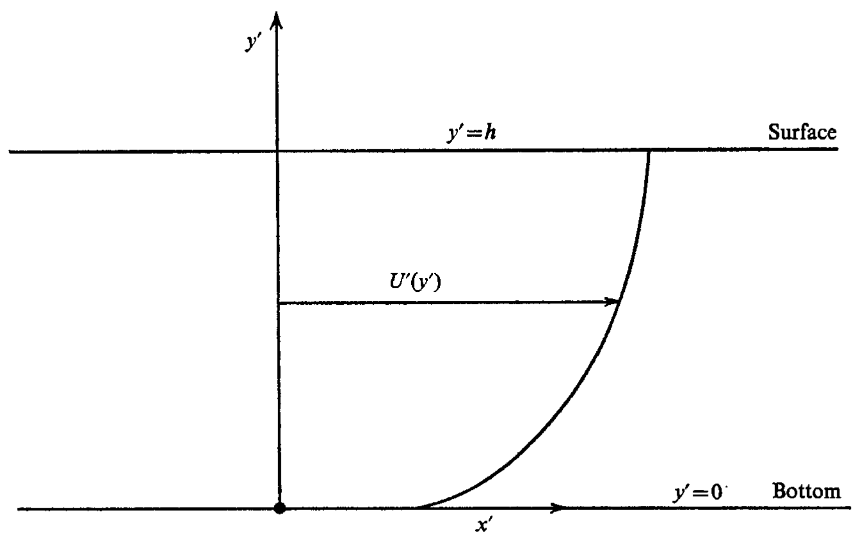


FIGURE 1. Basic flow.

If $u'$ and $v'$ are the components of velocity in the $x'$ and $y'$ directions the equations are

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+\frac{1}{\rho'}\frac{\partial p'}{\partial x'} = 0, \tag{2.1}$$

$$\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+\frac{1}{\rho'}\frac{\partial p'}{\partial y'}+g = 0. \tag{2.2}$$

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'} = 0, \tag{2.3}$$

where $\rho'$ is the (constant) density, $p'$ the pressure and $g$ the gravitational acceleration.

The boundary conditions on the surface, $y' = h+\eta'(x',t')$, are constant surface pressure

$$p'(x',h+\eta'(x',t'),t') = P_0' \text{ (say)}, \tag{2.4}$$

and continuity of surface velocity and fluid velocity

$$v'(x',h+\eta'(x',t'),t') = \frac{\partial \eta'}{\partial t'}+u'\frac{\partial \eta'}{\partial x'}. \tag{2.5}$$

On the bottom only one condition is required for an inviscid flow

$$v' = 0 \quad \text{on} \quad y' = 0. \tag{2.6}$$

Since the initial equations (2.1)–(2.3) contain only derivatives of $p$, a more convenient form of (2.4) is

$$\frac{\partial p'}{\partial x'} + \frac{\partial \eta'}{\partial x'}\frac{\partial p'}{\partial y'} = 0 \quad \text{on} \quad y' = h + \eta'. \tag{2.7}$$

Non-dimensionalization of these equations can be obtained by use of the scale parameters $L$, the length scale of a typical wave motion, $h$ the depth and a typical velocity $c = \sqrt{(gh)}$. The non-dimensional variables are chosen as follows:

$$x = \frac{x'}{L}, \quad y = \frac{y'}{h}, \quad t = \frac{ct'}{L}, \quad \eta = \frac{\eta'}{h}, \quad u = \frac{u'}{c}, \quad v = \frac{v'\delta}{c}, \quad p = \frac{p'}{\rho c^2}, \tag{2.8}$$

where $\delta = h/L$.

It will be observed that variations along the surface are assumed to occur on a length scale $L$ where as variations normal to the surface occur on a scale $h$. It has been found convenient to introduce the non-dimensional parameter $h/L$ into the scaling of the transverse velocity.

The equations (2.1)–(2.7) then reduce to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{v}{\delta^2}\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0, \tag{2.9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + \frac{v}{\delta^2}\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + 1 = 0, \tag{2.10}$$

$$\frac{\partial u}{\partial x} + \frac{1}{\delta^2}\frac{\partial v}{\partial y} = 0, \tag{2.11}$$

with

$$\left.\begin{aligned} \frac{\partial p}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial p}{\partial y} &= 0 \\ v &= \delta^2\left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}\right) \end{aligned}\right\} \quad \text{on} \quad y = 1 + \eta; \tag{2.12}$$

and

$$v = 0 \quad \text{on} \quad y = 0. \tag{2.13}$$

The approximation of long waves on shallow water then requires that $\delta \ll 1$. It should be noted that the scaling of $v'$ introduced into (2.8) shows that $v'$ is taken to be large in the theory.

The primary flow will be assumed to be of the form $u = U(y)$.

A wave motion of small amplitude defined by a parameter $\epsilon$ is now assumed to occur. We write

$$\left.\begin{aligned} u &= U + \epsilon\bar{u}, \\ v &= \epsilon^2\bar{v}, \\ p &= P + \epsilon\bar{p}, \\ \eta &= \epsilon\bar{\eta}, \end{aligned}\right\} \tag{2.14}$$

where $P = P_0 - (y-1)$ where $P_0 = P_0'/(\rho c^2)$.

$P$ is the hydrostatic pressure variation due to gravity suitably non-dimensionalized.

Two small parameters $\delta$ and $\epsilon$ have now been introduced. For a non-trivial limit $\delta \to 0$, $\epsilon \to 0$ it is now necessary to assume that

$$\delta^2 = 0(\epsilon) \quad \text{or} \quad \delta^2 = K\epsilon. \tag{2.15}$$

This may be written $a\lambda^2/h^3 = O(1)$ where $a$ is the amplitude and $\lambda$ the wavelength of the wave, a result recognized by Korteweg & de Vries (1895). It will now be observed that although $v'$ was assumed large the actual perturbation to $v'$ is order $\epsilon^{\frac{3}{2}}$ as opposed to a perturbation to the $u$ component of velocity relative to the undisturbed flow of magnitude $\epsilon$.

Substituting the new variable of (2.14) into the equations of motion (2.9)–(2.13) gives

$$\left.\begin{aligned} \frac{\partial \bar{u}}{\partial t} + U\frac{\partial \bar{u}}{\partial x} + \epsilon\bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\bar{v}}{K}U' + \epsilon\frac{\bar{v}}{K}\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{p}}{\partial x} &= 0, \\ \epsilon\left(\frac{\partial \bar{v}}{\partial t} + U\frac{\partial \bar{v}}{\partial x}\right) + \epsilon^2\left[\bar{u}\frac{\partial \bar{v}}{\partial x} + \frac{\bar{v}}{K}\frac{\partial \bar{v}}{\partial y}\right] + \frac{\partial \bar{p}}{\partial y} &= 0, \\ \frac{\partial \bar{u}}{\partial x} + \frac{1}{K}\frac{\partial \bar{v}}{\partial y} &= 0, \end{aligned}\right\} \tag{2.16}$$

with

$$\left.\begin{aligned} \frac{\partial \bar{p}}{\partial x} - \frac{\partial \bar{\eta}}{\partial x} + \epsilon\frac{\partial \bar{\eta}}{\partial x}\frac{\partial \bar{p}}{\partial y} &= 0, \\ \bar{v} = K\left[\frac{\partial \bar{\eta}}{\partial t} + U\frac{\partial \bar{\eta}}{\partial x} + \epsilon\bar{u}\frac{\partial \bar{\eta}}{\partial x}\right]&, \end{aligned}\right\} \quad \text{on} \quad y = 1 + \epsilon\bar{\eta}, \tag{2.17}$$

and

$$\bar{v} = 0 \quad \text{on} \quad y = 0.$$

No approximation has been introduced so far, but it is clear that all the non-linear terms in (2.16) and (2.17) are of order $\epsilon$. Henceforth $U'$ will be used to denote $dU/dy$.

## 3. Linearized theory

The small disturbance theory of Burns (1953) may now be readily deduced by taking the limit $\epsilon \to 0$ in (2.17). The resulting equations, which are linear, are

$$\left.\begin{aligned} \frac{\partial \bar{u}}{\partial t} + U\frac{\partial \bar{u}}{\partial x} + \frac{\bar{v}}{K}U' + \frac{\partial \bar{p}}{\partial x} &= 0, \\ \frac{\partial \bar{p}}{\partial y} &= 0, \\ \frac{\partial \bar{u}}{\partial x} + \frac{1}{K}\frac{\partial \bar{v}}{\partial y} &= 0, \end{aligned}\right\} \tag{3.1}$$

with

$$\left.\begin{aligned} \frac{\partial \bar{p}}{\partial x} &= \frac{\partial \bar{\eta}}{\partial x}, \\ \bar{v} &= K\left[\frac{\partial \bar{\eta}}{\partial t} + U\frac{\partial \bar{\eta}}{\partial x}\right], \end{aligned}\right\} \quad \text{on} \quad y = 1, \tag{3.2}$$

and

$$\bar{v} = 0 \quad \text{on} \quad y = 0.$$

Reduction of these equations to a single equation is complicated and not necessary for our purpose. We shall seek a wave motion propagation at a velocity $C$ only. For such a wave $\partial/\partial t = -C(\partial/\partial x)$ and the steady equations become

$$\left.\begin{aligned} (U-C)\frac{\partial \bar{u}}{\partial x}+\frac{\bar{v}}{K}U'+\frac{\partial \bar{\eta}}{dx} &= 0, \\ \frac{\partial \bar{u}}{\partial x}+\frac{1}{K}\frac{\partial \bar{v}}{\partial y} &= 0, \end{aligned}\right\} \tag{3.3}$$

with $$\bar{v} = K(U_1-C)(d\bar{\eta}/dx) \quad \text{on} \quad y=1, \tag{3.4}$$

and $$\bar{v} = 0 \quad \text{on} \quad y = 0,$$

where $$U_1 = U(1).$$

The pressure has been assumed constant with depth in accordance with the second of equations (3.1).

The continuity equation can be integrated to give

$$\frac{\bar{v}}{K} = -\int_0^y \frac{\partial \bar{u}}{\partial x}\,dy, \tag{3.5}$$

and the momentum equation can then be rewritten

$$(U-C)^2\frac{\partial}{\partial y}\left[\frac{1}{(U-C)}\int_0^y \frac{\partial \bar{u}}{\partial x}\,dy\right]+\frac{d\bar{\eta}}{dx} = 0. \tag{3.6}$$

Integration of this equation yields

$$\int_0^y \frac{\partial \bar{u}}{\partial x}\,dy = -\frac{d\bar{\eta}}{dx}(U-C)\int_0^y \frac{dy}{(U-C)^2}, \tag{3.7}$$

and differentiation gives

$$\bar{u} = -\bar{\eta}\frac{\partial}{\partial y}(U-C)\int_0^y \frac{dy}{(U-C)^2}. \tag{3.8}$$

This result will be written

$$\bar{u} = -\bar{\eta}\frac{\partial}{\partial y}(WI_2), \tag{3.9}$$

where $W = U-C$ and

$$I_n(y) = \int_0^y \frac{dy}{(U-C)^n}.$$

From (3.5),

$$\frac{\bar{v}}{K} = \bar{\eta}_x WI_2, \tag{3.10}$$

where $\bar{\eta}_x = d\bar{\eta}/dx$.

Comparing this with (3.4) shows that the propagation speed $C$ must satisfy

$$[I_2]_{y=1} = 1,$$

or $$\int_0^1 \frac{dy}{(U-C)^2} = 1. \tag{3.11}$$

This formula was originally derived by Burns (1953), where some of the properties of the propagation velocity $C$ are discussed. In particular, it may be noted that for $0 \leqslant U(0) < U(y) < U(1)$, $U''(y) < 0$, there are two values of $C$, one of which is greater than $U(1)$ and the other less than $U(0)$. The wave motion is now determined in terms of the arbitrary function $\eta(x)$.

We may expect that any attempt to improve upon this approximation will lead to a non-uniformity in the expansion scheme as in the case of $U \equiv 0$ which is a particular case of the present theory. To overcome this difficulty it is necessary to reconsider the theory in regions where $t = O(\epsilon^{-1})$.

## 4. The far field theory

In the far field $t = O(\epsilon^{-1})$, we introduce new time and space variables,

$$\bar{t} = \epsilon t, \tag{4.1}$$

and

$$\bar{x} = x - Ct, \quad \bar{y} = y, \tag{4.2}$$

where $C$ is the propagation velocity of the linearized theory as defined by (3.11). The variables are assumed to be of order unity so that time rates of change will be small compared with space rates of change following the linearized wave. The space and time derivatives become

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial \bar{t}} - C\frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}}. \tag{4.3}$$

Substituting in (2.16) gives

$$\left.\begin{aligned} (U-C)\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\bar{v}}{K}U' + \frac{\partial \bar{p}}{\partial \bar{x}} + \epsilon\left[\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{d\bar{u}}{d\bar{x}} + \frac{\bar{v}}{K}\frac{d\bar{u}}{d\bar{y}}\right] &= 0,\\ \frac{\partial \bar{p}}{\partial \bar{y}} + \epsilon(U-C)\frac{\partial \bar{v}}{\partial \bar{x}} + \epsilon^2\left[\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\bar{v}}{K}\frac{\partial \bar{v}}{\partial \bar{y}}\right] &= 0,\\ \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{K}\frac{\partial \bar{v}}{\partial \bar{y}} &= 0, \end{aligned}\right\} \tag{4.4}$$

with

$$\left.\begin{aligned} \frac{\partial \bar{p}}{\partial \bar{x}} - \frac{\partial \bar{\eta}}{\partial \bar{x}} + \epsilon\frac{\partial \bar{\eta}}{\partial \bar{x}}\frac{\partial \bar{p}}{\partial \bar{y}} &= 0,\\ \bar{v} = K\left[(U-C)\frac{\partial \bar{\eta}}{\partial \bar{x}} + \epsilon\left(\frac{\partial \bar{\eta}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{\eta}}{\partial \bar{x}}\right)\right]&, \end{aligned}\right\} \quad \text{on} \quad \bar{y} = 1 + \epsilon\bar{\eta}, \tag{4.5}$$

and

$$\bar{v} = 0 \quad \text{on} \quad \bar{y} = 0.$$

It is now obvious that the zeroth-order equations obtained by letting $\epsilon \to 0$ now correspond exactly with those derived in §3. The function $\bar{\eta}(x, t)$ is not however arbitrarily determined. To observe this it is necessary to go to a higher approximation. We formally expand the dependent variables as follows:

$$\left.\begin{aligned} \bar{u} &= u_0 + \epsilon u_1 + \ldots,\\ \bar{v} &= v_0 + \epsilon v_1 + \ldots,\\ \bar{p} &= p_0 + \epsilon p_1 + \ldots,\\ \bar{\eta} &= \eta_0 + \epsilon \eta_1 + \ldots, \end{aligned}\right\} \tag{4.6}$$

and obtain

$$\left.\begin{aligned}(U-C)\frac{\partial u_0}{\partial \bar{x}}+\frac{v_0}{K}U'+\frac{\partial p_0}{\partial \bar{x}}&=0,\\ \frac{\partial p_0}{\partial \bar{y}}&=0,\\ \frac{\partial u_0}{\partial \bar{x}}+\frac{1}{K}\frac{\partial v_0}{\partial \bar{y}}&=0,\end{aligned}\right\}\tag{4.7}$$

with

$$\left.\begin{aligned}\frac{\partial p_0}{\partial \bar{x}}-\frac{\partial \eta_0}{\partial \bar{x}}&=0,\\ v_0=K(U_1-C)\frac{\partial \eta_0}{\partial \bar{x}},\end{aligned}\right\}\quad \bar{y}=1,\tag{4.8}$$

and

$$v_0=0 \quad \text{on} \quad \bar{y}=0.$$

These are the equations of §3 and their solution may be written down in terms of the unknown function $\eta_0(\bar{x},\bar{t})$, as follows:

$$\left.\begin{aligned}u_0&=-\eta_0(\partial/\partial\bar{y})(WI_2),\\ v_0&=K\eta_{0\bar{x}}WI_2,\\ p_0&=\eta_0.\end{aligned}\right\}\tag{4.9}$$

The second-order equations are more complex, and may be written as follows:

$$\left.\begin{aligned}\frac{\partial u_0}{\partial \bar{t}}+\frac{\partial u_1}{\partial \bar{x}}(U-C)+u_0\frac{\partial u_0}{\partial \bar{x}}+\frac{v_0}{K}\frac{\partial u_0}{\partial \bar{y}}+\frac{v_1}{K}U'+\frac{\partial p_1}{\partial \bar{x}}&=0,\\ \frac{\partial p_1}{\partial \bar{y}}+(U-C)\frac{\partial v_0}{\partial \bar{x}}&=0,\\ \frac{\partial u_1}{\partial \bar{x}}+\frac{1}{K}\frac{\partial v_1}{\partial \bar{y}}&=0,\end{aligned}\right\}\tag{4.10}$$

with

$$\left.\begin{aligned}\frac{\partial p_1}{\partial \bar{x}}+\frac{\partial \eta_0}{\partial \bar{x}}\frac{\partial p_0}{\partial \bar{y}}&=0,\\ v_1+\eta_0\frac{\partial v_0}{\partial \bar{y}}&=K\left[\frac{\partial \eta_0}{\partial \bar{t}}+u_0\frac{\partial \eta_0}{\partial \bar{x}}+\frac{\partial \eta_1}{\partial \bar{x}}(U_1-C)+\eta_0\frac{\partial \eta_0}{\partial \bar{x}}U'\right],\end{aligned}\right\}\quad \text{on} \quad \bar{y}=1,\tag{4.11}$$

and

$$v_1=0 \quad \text{on} \quad \bar{y}=0.$$

The pressure may be determined explicitly from (4.10) as

$$p_1=\eta_1+K\eta_{0\bar{x}\bar{x}}\int_{\bar{y}}^{1}W^2I_2\,d\bar{y}.\tag{4.12}$$

Rewriting $\quad \dfrac{\partial u_1}{\partial \bar{x}}(U-C)+v_1\dfrac{U'}{K} \quad$ as $\quad -\dfrac{(U-C)^2}{K}\dfrac{\partial}{\partial \bar{y}}\left(\dfrac{v_1}{U-C}\right),$

the first equation of (4.10) may then be integrated from $y=0$ to $y=1$ to give

$$\frac{[v_1]_{\bar{y}=1}}{KW_1}+\eta_{0\bar{t}}\int_0^1\frac{1}{W^2}(WI_2)'\,d\bar{y}+\eta_0\eta_{0\bar{x}}\left[\int_0^1\left\{\frac{(WI_2)''}{W}I_2-\frac{(WI_2)'^2}{W^2}\right\}d\bar{y}\right]$$
$$-\eta_{1\bar{x}}-K\eta_{0\bar{x}\bar{x}\bar{x}}\int_0^1\int_y^1\int_0^{y_1}\frac{W^2(y_1)}{W^2(y)\,W^2(y_2)}\,dy_2\,dy_1\,dy=0.\tag{4.13}$$

Equation (4.11) may now be used with (4.13) to obtain

$$\eta_{0\bar{t}}\left[1+W_1\int_0^1\frac{(WI_2)'}{W^2}d\bar{y}\right]+\eta_0\eta_{0\bar{x}}\left[-\frac{2}{W_1}-2W_1'+W_1\int_0^1\frac{1}{W^2}\{(WI_2)''\,WI_2-(WI_2)'^2\}d\bar{y}\right]$$

$$-K\eta_{0\bar{x}\bar{x}\bar{x}}W_1\int_0^1\int_y^1\int_0^{y_1}\frac{W^2(y_1)}{W^2(y)\,W^2(y_2)}dy_2dy_1dy = 0. \quad (4.14)$$

The equation (4.14) is now solely an equation for the function $\eta_0(\bar{x},\bar{t})$. It may be simplified to give

$$2I_{31}\eta_{0\bar{t}}-3I_{41}\eta_0\eta_{0\bar{x}}-\eta_{0\bar{x}\bar{x}\bar{x}}JK = 0, \quad (4.15)$$

where $$I_{n1} = I_n(1),$$

and $$J = +\int_0^1\int_y^1\int_0^{y_1}\frac{W^2(y_1)}{W^2(y)\,W^2(y_2)}dy_2dy_1dy,$$

$$= +\int_0^1\int_0^{y_1}\frac{W^2(y_1)}{W^2(y)}dy_1dy-\int_0^1\int_0^{y}\int_0^{y_1}\frac{W^2(y_1)}{W^2(y)\,W^2(y_2)}dy_2dy_1dy.$$

Equation (4.15) is the Korteweg–de Vries equation (Cole 1968) with the coefficients modified to include the effect of shear. In the case of no basic flow $U \equiv 0$. Equation (3.11) gives $C = \pm 1$, and (4.15) reduces to

$$2\eta_{0\bar{t}}+3\eta_0\eta_{0\bar{x}}+\tfrac{1}{3}K\eta_{0\bar{x}\bar{x}\bar{x}} = 0, \quad (4.16)$$

which may be compared with the corrected form of that given by Cole (1968).

By introducing scaling factors and defining $\tau = \bar{t}/(-I_{31})$, $X = \bar{x}/(3JK)^{\frac{1}{2}}$, and $N = \eta_0 I_{41}/(3JK)^{\frac{1}{2}}$, (4.15) can be reduced to the form,

$$2N_\tau+3NN_X+\tfrac{1}{3}N_{XXX} = 0, \quad (4.17)$$

and the no flow results are reproduced exactly in these co-ordinates.

It should be noted that if the propagation speed $C$, which is greater than $U(1)$, is chosen then $I_{31}$ is necessarily negative and propagation occurs in the positive $x$ direction. If $C$ is such that $C < U(0)$, however, $I_{31}$ is positive and propagation occurs in the negative $x$ direction. This, of course, corresponds to propagation in the positive and negative $x$ directions in the no flow case.

## 5. Conclusions

The equation (4.15) may be reduced in the steady case of a wave propagating with velocity $c$ to

$$\eta_{0x}(3I_{41}\eta_0-2I_{31}c)+\eta_{0xxx}JK = 0. \quad (5.1)$$

Integrating once with respect to $x$ gives

$$\tfrac{3}{2}I_{41}\eta_0^2-2I_{31}c\eta_0+\eta_{0xx}JK = \tfrac{1}{2}A, \quad (5.2)$$

where $A$ is a constant. Multiplying by $\eta_{0x}$ and integrating once more gives

$$KJ\eta_{0x}^2 = 2I_{31}c\eta_0^2-I_{41}\eta_0^3+A\eta_0+B. \quad (5.3)$$

This is the familiar equation for cnoidal waves (Lamb 1953).

The solution of equation (5.3) may be written as

$$\eta_0 = v_2 + (v_3 - v_2)\,\mathrm{cn}^2\left[x\left(\frac{I_{41}(v_3 - v_1)}{4JK}\right)^{\frac{1}{2}}; \nu\right], \tag{5.4}$$

with

$$\nu = \frac{v_3 - v_2}{v_3 - v_1},$$

where cn denotes the Jacobian elliptic function and $v_1$, $v_2$, $v_3$ are the roots of the equation,

$$v^3 - \frac{2I_{31}c}{I_{41}}v^2 - \frac{A}{I_{41}}v - \frac{B}{I_{41}} = 0,$$

such that $v_1 < v_2 < v_3$.

For the particular case of a solitary wave for which $A = B = 0$,

$$\eta_{0x}^2 = \frac{\eta_0^2}{JK}[2I_{31}c - I_{41}\eta_0].$$

We see that the amplitude of the wave is

$$[\eta_0]_{\max} = \frac{2I_{31}c}{I_{41}}, \tag{5.5}$$

which relates the speed of propagation relative to the linearized wave speed and the wave amplitude. This is Benjamin's (1962) result.

It may be demonstrated that the scaling introduced in (4.17) reduces (5.4) to the standard result for the case of waves on a stationary surface. The complete theory of cnoidal waves thus applies to the case of waves on a shear flow and all that is necessary is the computation of the quantities $I_{31}$, $I_{41}$ and $J$ for the appropriate velocity distribution and wave speed. As noted in §4, the two wave speeds associated with the flow then give upstream and downstream propagation in a way directly analogous to that in the zero velocity case.

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